13.2 Calculus on 3D Curves

Example: Consider

$$x = t, y = 2 - t^2$$

which can also be written as

$$\mathbf{r}(t) = \langle t, 2 - t^2 \rangle$$

1. Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$. What do these represent?

2. When $t = 1$, what is the location and what is the slope of the tangent line?

3. Find a vector in the direction of the tangent line at $t = 1$.

$\boxed{1}$ $\frac{dx}{dt} = 1$ = horizontal velocity

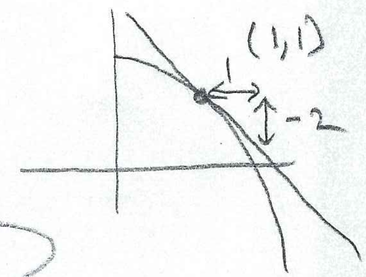
$\frac{dy}{dt} = -2t$ = vertical velocity

$\boxed{2}$ $t = 1 \Rightarrow x = 1, \; y = 1$

$x'(1) = 1, \quad y'(1) = -2$

RECALL

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2}{1}$$



$y = -2(x-1) + 1$

MATH 124

$\boxed{3}$ MATH 126

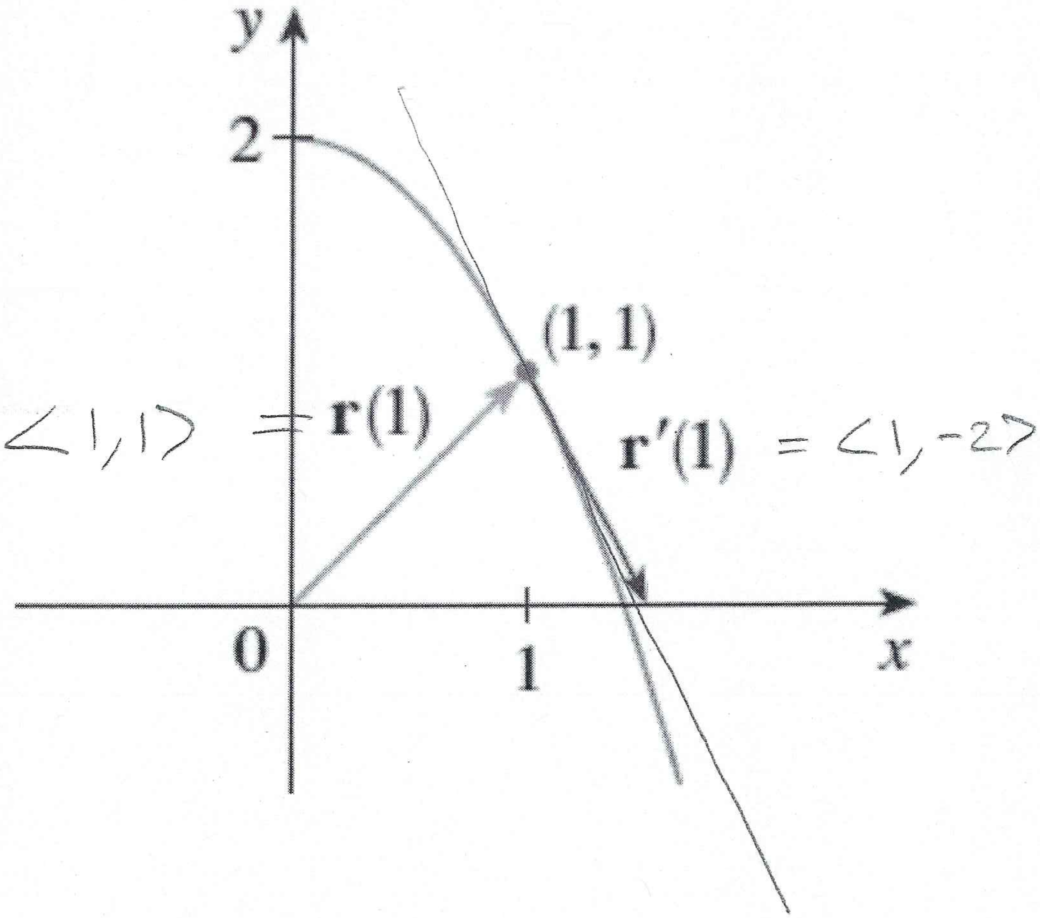
$\vec{r}_0 = \langle 1, 1 \rangle$

$\vec{v} = \langle 1, -2 \rangle$ = TANGENT VECTOR

$$\boxed{\begin{aligned} x &= 1 + 1t \\ y &= 1 + -2t \end{aligned}}$$

Visual of last example:

$$\mathbf{r}(t) = \langle t, 2 - t^2 \rangle$$



$$\boxed{\begin{aligned} x &= 1 + t \\ y &= 1 - 2t \end{aligned}}$$

TANGENT LINE

In general: Vector Calculus

$\vec{r}(t+h) - \vec{r}(t)$

For $\quad \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, we define

$$\vec{r}\,'(t) = \lim_{h \to 0} \left\langle \frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h}, \frac{z(t+h) - z(t)}{h} \right\rangle$$

which is the same as

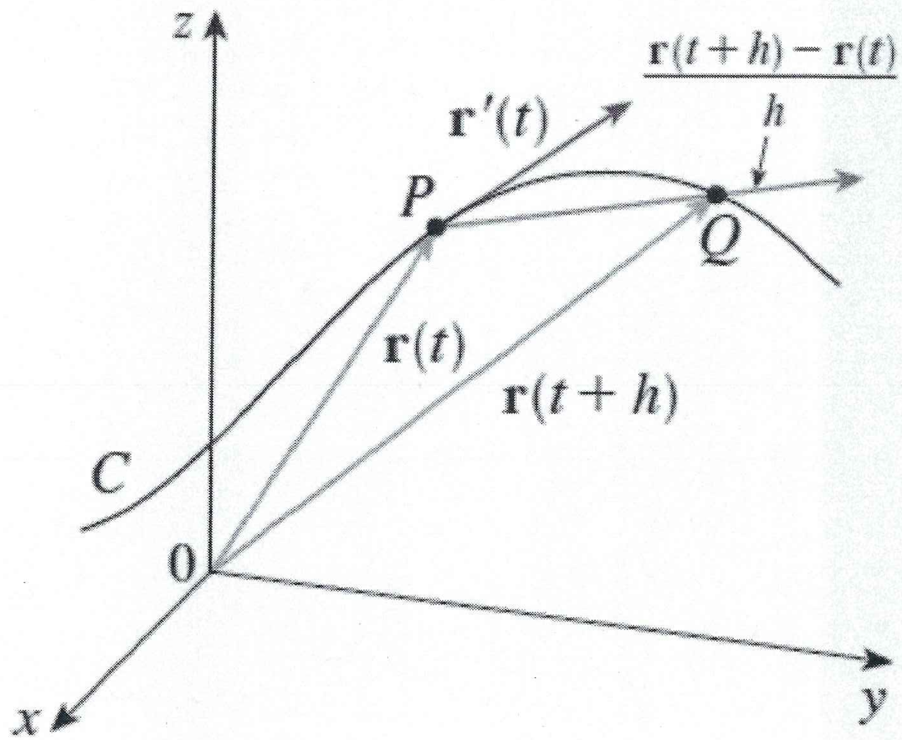
$$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

And

$$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

is a tangent vector to the curve.

Do calculus **component-wise**!



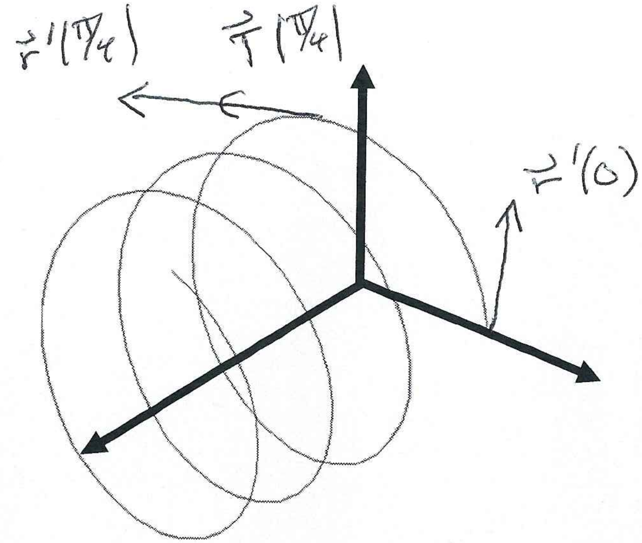
NOTE: ① $\vec{PQ} = \vec{r}(t+h) - \vec{r}(t)$

② DIVIDING BY h, RESCALES BUT KEEPS DIRECTION THE SAME.

Example

$\vec{r}(t) = \langle t, \cos(2t), \sin(2t) \rangle$.

1. Find $\vec{r}'(t)$.
2. Find $\vec{r}(0)$ and $\vec{r}(\pi/4)$.
3. Find $\vec{r}'(0)$ and $\vec{r}'(\pi/4)$.
4. Find the unit tangent vector $\vec{T}(t)$ at $t = \pi/4$.



$\boxed{1}$ $\vec{r}'(t) = \langle 1, -2\sin(2t), 2\cos(2t) \rangle$

$\boxed{2}$ $\vec{r}(0) = \langle 0, 1, 0 \rangle$ $\quad$ $\vec{r}(\pi/4) = \langle \pi/4, 0, 1 \rangle$

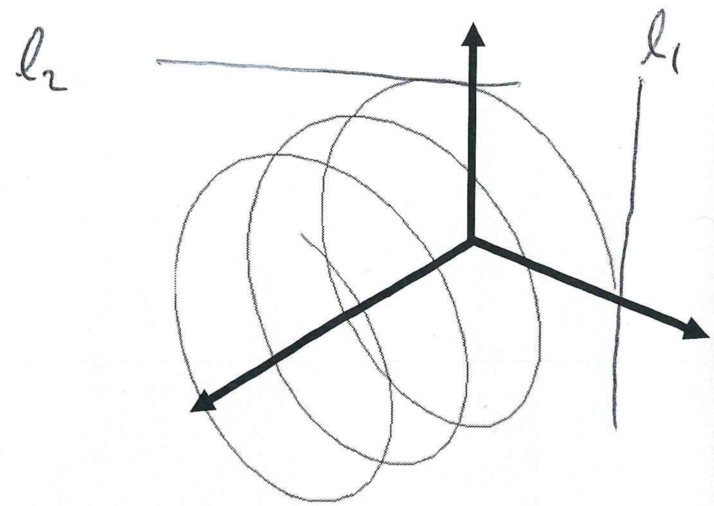
$\boxed{3}$ $\vec{r}'(0) = \langle \underbrace{1, 0, 2}_{\text{PARALLEL TO XZ-PLANE}} \rangle$ $\quad$ $\vec{r}'(\pi/4) = \langle \underbrace{1, -2, 0}_{\text{PARALLEL TO xy-PLANE}} \rangle$

$\boxed{4}$ $\vec{T}(\pi/4) = \dfrac{1}{|\vec{r}'(\pi/4)|} \vec{r}'(\pi/4) = \dfrac{1}{\sqrt{1+4+0}} \langle 1, -2, 0 \rangle = \left\langle \dfrac{1}{\sqrt{5}}, \dfrac{-2}{\sqrt{5}}, 0 \right\rangle$

Example Continued

$$\vec{r}(t) = \langle t, \cos(2t), \sin(2t) \rangle.$$

5. Find parametric equations for the tangent line at $t = 0$.
6. Find parametric equation for the tangent line at $t = \pi/4$.



$\boxed{5}$ ℓ_1

$x = 0 + t$
$y = 1 + 0t$
$z = 0 + 2t$

$\boxed{6}$ ℓ_2

$x = \pi/4 + t$
$y = 0 - 2t$
$z = 1 + 0t$

Example: Antiderivatives

First some review

Find the antiderivative of

$$f'(t) = \sin(t) + e^{2t} - \frac{t^3}{5}$$

with $f(0) = 7$.

$f(t) = \int \sin(t) + e^{2t} - \frac{1}{5}t^3 \, dt$

$f(t) = -\cos(t) + \frac{1}{2}e^{2t} - \frac{1}{20}t^4 + C$

$f(0) = 7 \Rightarrow -1 + \frac{1}{2} - 0 + C = 7$

$\Rightarrow C = 7 + \frac{1}{2} = \frac{15}{2}$

$$\boxed{f(t) = -\cos(t) + \frac{1}{2}e^{2t} - \frac{1}{20}t^4 + \frac{15}{2}}$$

CHECK!!!

Now find the antiderivative of

$$\vec{r}'(t) = \langle e^{3t}, t^4, \sin(t) - t \rangle.$$

with $\vec{r}(0) = \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$.

$\vec{r}(t) = \int \vec{r}'(t)\, dt$

$= \langle \frac{1}{3}e^{3t} + C_1, \frac{1}{5}t^5 + C_2, -\cos(t) - \frac{1}{2}t^2 + C_3 \rangle$

$\vec{r}(0) = \langle 1, 3, -2 \rangle \Rightarrow$ $\frac{1}{3} + C_1 = 1 \Rightarrow C_1 = 2/3$

$0 + C_2 = 3 \Rightarrow C_2 = 3$

$-1 + C_3 = -2 \Rightarrow C_3 = -1$

$$\boxed{\vec{r}(t) = \langle \frac{1}{3}e^{3t} + \frac{2}{3}, \frac{1}{5}t^5 + 3, -\cos(t) - \frac{1}{2}t^2 - 1 \rangle}$$